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The sum involving derivative of $\zeta(s)$ over simple zeros

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Abstract

In this paper, we study the lower bound for the sum of the absolute value of inverse of the derivative of the Riemann zeta-function running over it's zeros assuming only that all the zeros of $\zeta(s)$ are simple.

MSC: primary 11M; secondary 11M06

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1. Introduction

Throughout the paper, the general parameter T satisfies $T \geqslant T_0$ where T_0 is a sufficiently large positive number. For $m \geqslant 1$ an integer, $\zeta^{(m)}(s)$ denotes the mth derivative of $\zeta(s)$, $\rho = \beta + i\gamma$ denotes a nontrivial zero of $\zeta(s)$, $N^{(1)}(T)$ and $N(T)$ denote the

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number of simple and the total number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ in the rectangle $\{0 \leqslant \beta \leqslant 1, 0 < \gamma < T\}$, respectively. The Riemann hypothesis (hereafter we refer to this as RH) asserts that all the non-trivial complex zeros of $\zeta(s)$ are on the critical line $\Re s = \frac{1}{2}$.

On p. 374 of Titchmarsh's book [12, p. 374] revised by D.R. Heath-Brown, it is proved that the series

$$\sum \left|\rho\zeta^{(1)}(\rho)\right|^{-1} \tag{1.1}$$

diverges, assuming RH and that all the zeros of $\zeta(s)$ we simple.

In [7], Gonek studies the asymptotic formula for the quantity

$$\sum_{0<\gamma<T} \zeta^{(\mu)}(\rho + i\alpha L^{-1})\zeta^{(\nu)}(1 - \rho - i\alpha L^{-1}) \tag{1.2}$$

(with $L = \frac{1}{2\pi}\log\frac{T}{2\pi}$, $|\alpha| \leqslant \frac{L}{2}$). Assuming RH, one deduces from his asymptotic formula

$$\sum_{0<\gamma<T} \left|\zeta^{(\mu)}\left(\frac{1}{2} + i\gamma\right)\right|^2 \sim N(T)\left(\frac{\mu}{\mu+1}\right)^2 \left(\frac{1}{T}\int_0^T \left|\zeta^{(\mu)}\left(\frac{1}{2} + it\right)\right|^2 dt\right). \tag{1.3}$$

One can combine (1.3) for $\mu = 1$ with Heath-Brown's estimate [8] (see also [4,9])

$$N^{(1)}(T) \geqslant \tfrac{1}{3}N(T)$$

or even with the better estimate of Conrey [3]

$$N^{(1)}(T) \geqslant \tfrac{2}{5}N(T),$$

to obtain the estimate

$$\sum_{|\gamma| \leqslant T}^{*} \left|\rho\zeta^{(1)}(\rho)\right|^{-1} \gg (\log T)^{1/2} \tag{1.4}$$

assuming RH, where the star means that the summation is taken over the simple zeros only. Estimate (1.4) was improved in [5,6] to the unconditional result

$$\sum_{|\gamma| \leqslant T}^{*} \left|\rho\zeta^{(1)}(\rho)\right|^{-1} \gg (\log T)^{3/4}.$$

From the arguments of [5,6] it follows that

$$\sum_{|\gamma|\leqslant T}^{*} \left|\zeta^{(1)}(\rho)\right|^{-1} \gg T(\log T)^{-1/4}.$$

The goal of this paper is to prove

Theorem 1. *If all the zeros of $\zeta(s)$ are simple, then the estimate*

$$\sum_{|\gamma|\leqslant T} \left|\zeta^{(1)}(\rho)\right|^{-1} \gg T \tag{1.5}$$

holds and we do not need the Riemann Hypothesis to uphold the above inequality.

To remove RH in Theorem 1, we use a deep result of Ramachandra and Sankaranarayanan (see [10]) which we state as a Lemma in Section 2. We are able to establish a slightly weaker form of Theorem 1 without invoking any deep zero density results. We prove

Theorem 2. *If all the zeros of $\zeta(s)$ are simple, then we have*

$$\sum_{|\gamma|\leqslant T} \left|\zeta^{(1)}(\rho)\right|^{-1} = \Omega_{+}(T). \tag{1.6}$$

Our original proof of Theorem 1 was somehow lengthy and relatively complicated. The referee kindly informed authors of the unpublished argument of Montgomery in proving Theorem 1 of our paper under RH. Montgomery's argument is short and powerful. In this paper we incorporate his argument to give a short and direct proof of Theorems 1 and 2.

The main idea of the paper can be explained as follows. We can obtain Theorem 1 by considering the integral

$$\frac{1}{2\pi i}\int_{c_1-iT}^{c_1+iT} F(s)(\zeta(s))^{-1}\,ds \tag{1.7}$$

for a suitable test function $F(s)$ where c_1 is any positive real number such that $\left|\frac{F(c_1+it)}{\zeta(c_1+it)}\right| \ll 1$. Montgomery chooses $F(s)=1$ whereas we chose $F(s)=(x^s-y^s)/s$ with x, y specific numbers in the earlier version. Then, we used Perron's formula and moved the line of integration in (1.7) to the far left taking into account the contributions of the trivial zeros of $\zeta(s)$ too. At the end, we showed that the contributions coming from the trivial zeros of $\zeta(s)$ could be ignored as long as we estimate the lower bound to the sum in the left-hand side of (1.5).

Thus our earlier proof is essentially the same as Montgomery's. The main advantage in choosing $F(s) = 1$ is that we can argue in a suitable rectangular contour in the upper half-plane $(t \geqslant 1)$ with the left vertical line being $\sigma = -1$, the right vertical line being $\sigma = 2$. Thus we can completely ignore the trivial zeros of $\zeta(s)$ and avoid the use of Perron's formula in proving Theorem 1. Hence, the proof presented in Section 3 is direct and simple. On the other hand, the proof of Theorem 2 uses a different test function, namely $F(s) = e^{\frac{s^2}{T^2}}$ and only a very minor zero density estimate of $\zeta(s)$.

Remark. In a series of papers (see [1,2,11]), the singularities of some special type of functions involving translates of zeta-functions have been investigated. Sometimes even local theorems about the poles of these functions have been established.

2. Auxiliary lemma

The following lemma is crucial in removing RH in the above mentioned Theorem 1.

Lemma. *Let $H = T^{\frac{1}{3}}$. Then, unconditionally we have*

$$\min_{T \leqslant t \leqslant T+H} \max_{\frac{1}{2} \leqslant \sigma \leqslant 2} |\zeta(\sigma + it)|^{-1} < \exp(C(\log\log T)^2)$$

where C is an absolute positive constant.

Proof. This is a part of Theorem 2 of [10]. □

The above lemma is a variant of Theorem 14.16 (due to Littlewood) of [12], where a similar result is proved under RH. It should be mentioned that in [10] a zero density estimate for $\zeta(s)$ in short intervals has been used to remove RH and thus they obtained the above lemma unconditionally.

3. Proof of Theorem 1

From now on, we assume that all the non-trivial zeros $\rho = \beta + i\gamma$ (with $0 < \beta < 1$) of $\zeta(s)$ are simple. We note that these zeros are symmetrically situated with respect to the real axis and $|\gamma| > 10$. From the lemma, we immediately observe that we can choose a point $T_1 \in (\frac{T}{2}, T)$ such that

$$\max_{\frac{1}{2} \leqslant \sigma \leqslant 2} |\zeta(\sigma + iT_1)|^{-1} \ll \exp(C(\log\log T)^2). \tag{3.1}$$

From the functional equation, we have

$$\begin{aligned} Q_1 &=: \max_{-1 \leqslant \sigma \leqslant \frac{1}{2}} |\zeta(\sigma + iT_1)|^{-1} \\ &\ll \max_{-1 \leqslant \sigma \leqslant \frac{1}{2}} |T_1|^{\sigma - \frac{1}{2}} |\zeta(1 - \sigma - iT_1)|^{-1} \\ &\ll \max_{\frac{1}{2} \leqslant 1-\sigma \leqslant 2} |\zeta(1 - \sigma - iT_1)|^{-1} \\ &\ll \exp(C(\log\log T)^2). \end{aligned} \tag{3.2}$$

Now, we combine (3.1) with (3.2) and obtain for the chosen $T_1 \in (\frac{T}{2}, T)$ the inequality

$$\max_{-1 \leqslant \sigma \leqslant 2} |\zeta(\sigma + iT_1)|^{-1} \ll \exp(C(\log\log T)^2). \tag{3.3}$$

Let $\mathcal{C}$ denote the rectangular contour obtained by joining the vertices $2+i$, $2+iT_1$, $-1+iT_1$, $-1+i$ and $2+i$ with line segments in the anti-clockwise direction. We start directly with the contour integral

$$\frac{1}{2\pi i} \int_{\mathcal{C}} (\zeta(s))^{-1}\, ds = \sum_{0<\gamma<T_1} \left(\zeta^{(1)}(\rho)\right)^{-1}, \tag{3.4}$$

on computing the residues of the integrand at $s = \rho$. Now, we observe that from the functional equation, we have $|\zeta(-1+it)| \asymp |t|^{\frac{3}{2}}$ for $|t| \geqslant 1$ and hence, the left vertical line contribution in absolute value is

$$\left| \frac{1}{2\pi i} \int_{-1+i}^{-1+iT_1} (\zeta(s))^{-1}\, ds \right| \ll \int_1^{\infty} t^{-\frac{3}{2}}\, dt \ll 1. \tag{3.5}$$

From (3.3), we note that the top horizontal line contribution in absolute value is

$$\left| \frac{1}{2\pi i} \int_{-1+iT_1}^{2+iT_1} (\zeta(s))^{-1}\, ds \right| \ll \exp(C(\log\log T)^2) \tag{3.6}$$

and the bottom horizontal line contribution in absolute value is,

$$\left| \frac{1}{2\pi i} \int_{-1+i}^{2+i} (\zeta(s))^{-1}\, ds \right| \ll 1. \tag{3.7}$$

The right hand vertical line contribution is

$$\begin{aligned} Q_2 &=: \frac{1}{2\pi i}\int_{2+i}^{2+iT_1}(\zeta(s))^{-1}\,ds \\ &= \frac{1}{2\pi}\int_1^{T_1}\left\{1+\sum_{n=2}^{\infty}\frac{\mu(n)}{n^{2+it}}\right\}dt \\ &= \frac{1}{2\pi}(T_1-1)+O\left(\sum_{n=2}^{\infty}\frac{|e^{iT_1\log n}-e^{i\log n}|}{n^2\log n}\right) \\ &= \frac{T_1}{2\pi}+O(1). \end{aligned} \tag{3.8}$$

From (3.4)–(3.8), we get

$$\sum_{0<\gamma<T_1}\left(\zeta^{(1)}(\rho)\right)^{-1}=\frac{T_1}{2\pi}+O(\exp(C(\log\log T)^2)). \tag{3.9}$$

Thus, taking absolute values on both sides of (3.9), we obtain

$$\sum_{|\gamma|<T}\left|\zeta^{(1)}(\rho)\right|^{-1}\geqslant\left|\sum_{0<\gamma<T_1}\left(\zeta^{(1)}(\rho)\right)^{-1}\right|\gg T.$$

This proves Theorem 1.

4. Proof of Theorem 2

We assume that all the non-trivial zeros of $\zeta(s)$ are simple. We start with the integral

$$I=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}e^{s^2/T^2}(\zeta(s))^{-1}\,ds. \tag{4.1}$$

Expanding out the series, we obtain

$$I=\sum_{n=1}^{\infty}\mu(n)\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}e^{s^2/T^2}n^{-s}\,ds. \tag{4.2}$$

The inner integral is

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} e^{s^2/T^2} n^{-s}\, ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-u^2/T^2} e^{-iu\log n}\, du$$
$$= \frac{T}{2\pi}\hat{\phi}\left(\frac{T}{2\pi}\log n\right), \tag{4.3}$$

where $\phi(u) = e^{-u^2}$. Note that $\hat{\phi}(\xi) = \sqrt{\pi}e^{-\pi^2\xi^2}$ and

$$I = \frac{T}{2\pi}\left(\sqrt{\pi} + \sqrt{\pi}\sum_{n\geqslant 2}\mu(n)e^{-T^2(\log n)^2/4}\right). \tag{4.4}$$

The sum in (4.4) is

$$\ll \sum_{n\geqslant 2} e^{-T^2(\log n)^2/4} \ll \int_2^{\infty} e^{-T^2(\log u)^2/4}\, du = T^{-1}\int_{\frac{\log 2}{2}T}^{\infty} e^{-x^2+2x/T}\, dx$$
$$\ll T^{-1} \tag{4.5}$$

and hence

$$I = \sqrt{\pi}\,\frac{T}{2\pi} + O(1). \tag{4.6}$$

Moving the integral in (4.1) to the line $\Re(s) = -1$ we get

$$I = \sum_{\rho} e^{\rho^2/T^2}\left(\zeta^{(1)}(\rho)\right)^{-1} + \frac{1}{2\pi i}\int_{-1-i\infty}^{-1+i\infty} e^{s^2/T^2}(\zeta(s))^{-1}\, ds, \tag{4.7}$$

where the sum in the right-hand side of (4.7) runs over all the non-trivial zeros ρ of $\zeta(s)$. Note that we may justify this contour move by the following reasoning. It is well known that for sufficiently large V (see for example [12, Theorem 9.2])

$$N(V+1) - N(V) \ll \log V.$$

Therefore, we can divide the interval $[V, V+1]$ into $O(\log V)$ abutting sub-intervals of width $O(\frac{1}{\log V})$ and then choose an $U \in [V, V+1]$ such that the distance from U to the nearest ordinate of ρ would be $\gg \frac{1}{\log U}$. For such U, on the line $\Im(s) = U$ with $-1 \leqslant \sigma \leqslant 2$, we have uniformly (see for example [12, Theorem 9.6(A)])

$$\frac{\zeta^{(1)}(\sigma + iU)}{\zeta(\sigma + iU)} \ll \log^2 U. \tag{4.8}$$

Integrating this with respect to σ (for $-1 \leqslant \sigma \leqslant 2$) yields $\log \zeta(\sigma + iU) \ll \log^2 U$ and hence

$$\left|(\zeta(\sigma + iU))^{-1}\right| \ll e^{c_2(\log U)^2} \tag{4.9}$$

uniformly for $-1 \leqslant \sigma \leqslant 2$ (see also [12, Theorem 9.7]) where even a better estimate is proved). Therefore the integrand on horizontal lines of this type are bounded by

$$e^{-U^2/T^2 + c_2(\log U)^2} \ll 1, \tag{4.10}$$

provided $U \geqslant T(\log T)^2$. This justifies the contour shift. Now on the line $\Re(s) = -1$ we have $\zeta(s) \asymp u^{\frac{3}{2}}$ where $s = -1 + iu$ and hence the left vertical line integral is in absolute value

$$\ll \int_{-\infty}^{\infty} e^{-u^2/T^2} \min(1, u^{-\frac{3}{2}})\, du \ll 1. \tag{4.11}$$

Therefore

$$\sum_{\rho} e^{\rho^2/T^2} \left(\zeta^{(1)}(\rho)\right)^{-1} = \sqrt{\pi}\frac{T}{2\pi} + O(1), \tag{4.12}$$

with the sum in the left-hand side of (4.12) runs over all the non-trivial zeros of $\zeta(s)$. By taking absolute values in (4.12), it follows that

$$T \ll \sum_{\gamma > 10} e^{-\gamma^2/T^2} \left|\zeta^{(1)}(\rho)\right|^{-1} = \frac{2}{T^2} \int_{10}^{\infty} J_{-1/2}(u) u e^{-u^2/T^2}\, du, \tag{4.13}$$

where

$$J_{-1/2}(u) =: \sum_{10 < \gamma \leqslant u} \left|\zeta^{(1)}(\rho)\right|^{-1}.$$

Now assume $J_{-1/2}(u) = o(u)$ or $J_{-1/2}(u) \leqslant \varepsilon u$ if $u \geqslant c_3$. Then we have

$$T \ll T^{-2} + \varepsilon \int_{c_3}^{\infty} \frac{u^2}{T^2} e^{-u^2/T^2}\, du = T^{-2} + 2\varepsilon T \int_{\frac{c_3}{T}}^{\infty} x^2 e^{-x^2}\, dx. \tag{4.14}$$

Thus we have

$$T \leqslant c_4 \varepsilon T \tag{4.15}$$

for T sufficiently large. However, we may choose ε sufficiently small and we have a contradiction. Therefore we conclude that $J_{-1/2}(T) = \Omega_+(T)$. This proves Theorem 2.

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